

APPORTIONMENT

Topics in Contemporary Mathematics

MA 103

Summer II, 2013

Content

- ✧ **Apportionment Problems**
- ✧ **Hamilton's Method**
- ✧ **The Alabama and Other Paradoxes**
- ✧ **Jefferson's Method**
- ✧ **Adam's Method**
- ✧ **Webster's Method**

Apportionment Problems

There are two critical elements in the dictionary definition of the word *apportion* :

- (1) We are dividing and assigning things, and
- (2) we are doing this on a proportional basis and in a planned, organized fashion.

In the next slide we will look at an example that illustrates the nature of the problem we are dealing with.

Example: Kitchen Capitalism

Mom has a total of 50 identical pieces of candy, which she is planning to divide among her five children (this is the *division* part). She wants to teach her children about the value of work and about the relationship between work and reward.

She announces to the kids that the candy is going to be divided at the end of the week in proportion to the amount of time each of them spends helping with the weekly kitchen chores—if you worked twice as long as your brother you get twice as much candy, and so on (this is the *due and proper proportion* part).

Unwittingly, mom has turned this division problem into an apportionment problem.

Example: Kitchen Capitalism

At the end of the week, the numbers are in. The Table shows the amount of work done by each child during the week.

Amount of Work (in Minutes) per Child

Child	Alan	Betty	Connie	Doug	Ellie	Total
Minutes worked	150	78	173	204	295	900

Example: Kitchen Capitalism

According to the ground rules, Alan, who worked 150 out of a total of 900 minutes, should get $8 \frac{1}{3}$ pieces.

Here comes the problem: Since the pieces of candy are indivisible, it is impossible for Alan to get his pieces—he can get 8 pieces (and get shorted) or he can get 9 pieces (and someone else will get shorted).

Example: Kitchen Capitalism

A similar problem occurs with each of the other children.

Betty's exact fair share should be $4 \frac{1}{3}$ pieces;

Connie's should be $9 \frac{11}{18}$ pieces;

Doug's, $11 \frac{1}{3}$ pieces; and

Ellie's, $16 \frac{7}{18}$ pieces.

Because none of these shares can be realized, an absolutely fair apportionment of the candy is going to be impossible.

What should mom do?

Apportionment Problem

The example shows all the elements of an apportionment problem – there are objects to be divided (the pieces of candy), and there is a proportionality criterion for the division (number of minutes worked during the week).

We will say that the pieces of candy are apportioned to the kids, and we will describe the final solution as an apportionment (Alan's apportionment is x pieces, Betty's apportionment is y pieces, etc.).

Apportionment: Basic Concepts and Terminology

Apportionment problems can arise in a variety of real-life applications –
dividing candy among children,
assigning nurses to shifts,
assigning buses to routes, and so on.

But the *gold standard for apportionment applications is the allocation of seats in a legislature*, and thus it is standard practice to borrow the terminology of legislative apportionment and apply it to apportionment problems in general.

Apportionment: Basic Concepts and Terminology

The basic elements of every apportionment problem are:

- the “states”,
- the “seats”, and
- the populations”

The “states”

This is the term we will use to *describe the parties having a stake* in the apportionment. Unless they have specific names (Azucar, Bahia, etc.), we will use A_1, A_2, \dots, A_N to denote the states.

Apportionment: Basic Concepts and Terminology

The "seats"

This term *describes the set of M identical, indivisible objects that are being divided among the N states.*

For convenience, we will assume that there are more seats than there are states, thus ensuring that every state can potentially get a seat. (This assumption does not imply that every state must get a seat!)

Apportionment: Basic Concepts and Terminology

The “populations”

This is a set of N positive numbers (for simplicity we will assume that they are whole numbers) *that are used as the basis for the apportionment of the seats to the states.*

We will use p_1, p_2, \dots, p_N to denote the state's respective populations and P to denote the total population $P = p_1 + p_2 + \dots + p_N$.

Apportionment: Basic Concepts and Terminology

Two of the most important concepts of the chapter are the *standard divisor* and the *standard quotas*.

We can now formally define these concepts using our new terminology and notation.

The standard divisor (SD)

This is the ratio of population to seats. It gives us a unit of measurement for our apportionment calculations.

Apportionment: Basic Concepts and Terminology

The standard quotas

The standard quota of a state is the *exact fractional number of seats that the state would get if fractional seats were allowed.*

We will use the notation q_1, q_2, \dots, q_N to denote the standard quotas of the respective states.

To find *a state's standard quota*, we divide the state's population by the *standard divisor*.

In general, the standard quotas can be expressed as fractions or decimals—round them to two or three decimal places.

Apportionment: Basic Concepts and Terminology

Upper and lower quotas

The **lower quota** is the standard quota *rounded down* and the **upper quota** is the standard quota *rounded up*.

In the unlikely event that the standard quota is a whole number, the lower and upper quotas are the same.

We will use L 's to denote lower quotas and U 's to denote upper quotas. For example, the standard quota $q_1 = 32.92$ has lower quota $L_1 = 32$ and upper quota $U_1 = 33$.

Apportionment Methods

Our main goal is to discover a “good” **apportionment method**—a reliable procedure that

1. will always produce a valid apportionment (exactly M seats are apportioned) and
2. will always produce a “fair” apportionment.

We will discuss several different methods and find out what is good and bad about each one.

Hamilton's Method

Every state gets at least its lower quota.

As many states as possible get their upper quota, with the one with highest residue (i.e., fractional part) having first priority, the one with second highest residue second priority, and so on.

Hamilton's Method

A little more formally, it goes like this:

- Step 1** Calculate each state's standard quota.
- Step 2** Give to each state its *lower quota*.
- Step 3** Give the surplus seats (one at a time) to the states with the largest *residues* (fractional parts) until there are no more surplus seats.

Example: Parador's Congress (Hamilton's Method)

Parador is a small republic in Central America and consist of six states: Azucar, Bahia, Cafe, Diamante, Esmeralda, and Felicidad (A, B, C, D, E, and F for short).

There are 250 seats in the Congress, which, according to the laws of Parador, are to be apportioned among the six states in proportion to their respective populations.

What is the "*correct*" apportionment?

Example: Parador's Congress (Hamilton's Method)

Parador's Congress: Hamilton Apportionment

State	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	Total
Standard quota	32.92	138.72	3.08	41.82	13.70	19.76	250.00
Lower quota	32	138	3	41	13	19	246
Residue	0.92	0.72	0.08	0.82	0.70	0.76	4.00
Order of Surplus	First	Last		Second		Third	
Apportionment	33	139	3	42	13	20	250

Hamilton's method (also known as Vinton's method or the method of largest remainders) was used in the United States only between 1850 and 1900.

Hamilton's Method

At first glance, Hamilton's method appears to be quite fair.

It could be reasonably argued that Hamilton's method has a major flaw in the way it relies entirely on the size of the residues without consideration of what those residues represent as a percent of the state's population.

In so doing, *Hamilton's method creates a systematic bias in favor of larger states over smaller ones.*

Hamilton's Method

A good apportionment method should be population neutral, meaning that it should not be biased in favor of large states over small ones or vice versa.

Hamilton's method has two important things going for it:

- (1) It is very easy to understand, and
- (2) it satisfies an extremely important requirement for fairness called the *quota rule*.

QUOTA RULE

No state should be apportioned a number of seats smaller than its lower quota or larger than its upper quota. (*When a state is apportioned a number smaller than its lower quota, we call it a **lower-quota violation**; when a state is apportioned a number larger than its upper quota, we call it an **upper-quota violation**.*)

Hamilton's Method & the Quota Rule

An apportionment method that guarantees that every state will be apportioned either its lower quota or its upper quota is said to satisfy the quota rule.

It is not hard to see that Hamilton's method satisfies the quota rule: Step 2 of Hamilton's method hands out to each state its lower quota. Right off the bat this guarantees that there will be no lower-quota violations.

In Step 3 some states get one extra seat, some get none; no state can get more than one. This guarantees that there will be no upper-quota violations.

Alabama Paradox – Hamilton's Method

The most serious flaw of Hamilton's method is commonly known as the **Alabama paradox**.

In essence, the Alabama paradox occurs when an increase in the total number of seats being apportioned, in and of itself, forces a state to lose one of its seats.

The best way to understand what this means is to look carefully at the following example.

Example: More Seats Means Fewer Seats

The small country of Calavos consists of three states: Bama, Tecos, and Ilnos. With a total population of 20,000 and 200 seats in the House of Representatives, the apportionment of the 200 seats under Hamilton's method is shown on the next slide.

Now imagine that overnight the number of seats is increased to 201, but nothing else changes. Since there is one more seat to give out, the apportionment has to be recomputed and is shown on the next slide.

Example: More Seats Means Fewer Seats

Hamilton Apportionment for $M = 200$ ($SD = 100$)

State	Population	Quota	Lower quota	Surplus	Apportionment
Bama	940	9.4	9	First	10
Tecos	9030	90.3	90	0	90
Ilnos	10,030	100.3	100	0	100
Total	20,000	200.0	199	1	200

Hamilton Apportionment for $M = 201$ ($SD \approx 99.5$)

State	Population	Quota	Lower quota	Surplus	Apportionment
Bama	940	9.45	9	0	9
Tecos	9030	90.75	90	Second	91
Ilnos	10,030	100.80	100	First	101
Total	20,000	201.00	199	2	201

Notice that for $M = 200$, the SD is 100; for $M = 201$, the SD drops to 99.5.)

Example: More Seats Means Fewer Seats

The shocking part of this story is the fate of Bama, the “little guy”. When the House of Representatives had 200 seats, Bama got 10 seats, but when the number of seats to be divided increased to 201, Bama’s apportionment went down to 9 seats. How did this paradox occur?

Notice the effect of the increase in M on the size of the residues: In a House with 200 seats, Bama is at the head of the priority line for surplus seats, but when the number of seats goes up to 201, Bama gets shuffled to the back of the line.

Arithmetic of the Alabama Paradox

The example illustrates the quirk of arithmetic behind the Alabama paradox: When we increase the number of seats to be apportioned, each state's standard quota goes up, but not by the same amount.

As the residues change, some states can move ahead of others in the priority order for the surplus seats. This can result in some state or states losing seats they already had.

The Population Paradox

A state could potentially lose some seats because its population got too big!. This phenomenon is known as *the population paradox*.

To be more precise, the **population paradox** occurs when state A *loses* a seat to state B even though the population of A grew at a higher rate than the population of B .

The next example will shed more light on this.

Example: A Tale of Two Planets

In the year 2525 the five planets in the Utopia galaxy finally signed a peace treaty and agreed to form a Federation governed by an Intergalactic Congress.

This is the story of the two apportionments that broke up the Federation.

Example: A Tale of Two Planets

Part I. The Apportionment of 2525.

The first Intergalactic Congress was apportioned using Hamilton's method, based on the population figures (in billions) shown in the second column of the Table. There were 50 seats apportioned.

Intergalactic Congress: Apportionment of 2525

Planet	Population	St. quota	Lower quota	Surplus	Apportionment
Alanos	150	$8.\bar{3}$	8	0	8
Betta	78	$4.\bar{3}$	4	0	4
Conii	173	$9.6\bar{1}$	9	First	10
Dugos	204	$11.\bar{3}$	11	0	11
Ellisium	295	$16.3\bar{8}$	16	Second	17
Total	900	50.00	48	2	50

Example: A Tale of Two Planets

The total population of the galaxy is 900 billion. After the lower quotas are handed out (column 4), there are two surplus seats. The first surplus seat goes to Conii and the other one to Ellisium. The last column shows the apportionments.

Intergalactic Congress: Apportionment of 2525

Planet	Population	St. quota	Lower quota	Surplus	Apportionment
Alanos	150	$8.\bar{3}$	8	0	8
Betta	78	$4.\bar{3}$	4	0	4
Conii	173	$9.6\bar{1}$	9	First	10
Dugos	204	$11.\bar{3}$	11	0	11
Ellisium	295	$16.3\bar{8}$	16	Second	17
Total	900	50.00	48	2	50

Keep an eye on the apportionments of Betta and Ellisium—they are central to how this story unfolds.

Example: A Tale of Two Planets

Part II. The Apportionment of 2535.

After 10 years of peace, there were a few changes in the planets' populations—an 8 billion increase in the population of Conii, and a 1 billion increase in the population of Ellisium. The populations of the other planets remained unchanged from 2525. Nonetheless, a new apportionment was required.

Intergalactic Congress: Apportionment of 2535

Planet	Population	St. quota	Lower quota	Surplus	Apportionment
Alanos	150	8.25	8	0	8
Betta	78	4.29	4	Second	5
Conii	181	9.96	9	First	10
Dugos	204	11.22	11	0	11
Ellisium	296	16.28	16	0	16
Total	909	50.00	48	2	50

Notice that the total population increased to 909 billion, so the standard divisor for this apportionment was $SD = 909/50 = 18.18$.

Example: A Tale of Two Planets

Intergalactic Congress: Apportionment of 2525

Planet	Population	St. quota	Lower quota	Surplus	Apportionment
Alanos	150	8.3	8	0	8
Betta	78	4.3	4	0	4
Conii	173	9.61	9	First	10
Dugos	204	11.3	11	0	11
Ellisium	295	16.38	16	Second	17
Total	900	50.00	48	2	50

Intergalactic Congress: Apportionment of 2535

Planet	Population	St. quota	Lower quota	Surplus	Apportionment
Alanos	150	8.25	8	0	8
Betta	78	4.29	4	Second	5
Conii	181	9.96	9	First	10
Dugos	204	11.22	11	0	11
Ellisium	296	16.28	16	0	16
Total	909	50.00	48	2	50

The one remarkable thing about the 2535 apportionment is that Ellisium lost a seat while its population went up and Betta gained that seat while its population remained unchanged!

Hamilton's Method-Population Paradox

The example illustrates a fundamental paradox: Under Hamilton's method, it is possible for a state with a positive population growth rate to lose one (or more) of its seats to another state with a smaller (or zero) population growth rate.

Once again, Hamilton's reliance on the residues to allocate the surplus seats is its undoing.

The New-States Paradox

The addition of a new state with its fair share of seats can, in and of itself, affect the apportionments of other states. This phenomenon is called the new-states paradox.

We will illustrate this with an example.

Example: Garbage Time

The Metro Garbage Company has a contract to provide garbage collection and recycling services in two districts of Metropolis, Northtown (with 10,450 homes) and the much larger Southtown (89,550 homes).

The company runs 100 garbage trucks, which are apportioned under Hamilton's method according to the number of homes in the district. A quick calculation shows that the standard divisor is $SD = 1000$ homes, a nice, round number which makes the rest of the calculations easy.

Example: Garbage Time

As a result of the apportionment, 10 garbage trucks are assigned to service Northtown and 90 garbage trucks to service Southtown.

Metro Garbage Truck Apportionments

District	Homes	Quota	Apportionment
Northtown	10,450	10.45	10
Southtown	89,550	89.55	90
Total	100,000	100.00	100

Example: Garbage Time

Now imagine that the Metro Garbage Company is bidding to expand its territory by adding the district of Newtown (5250 homes) to its service area.

In its bid to the City Council the company promises *to buy five additional garbage trucks for the Newtown* run so that its service to the other two districts is not affected.

But when the new calculations are carried out, there is a surprise: One of the garbage trucks assigned to Southtown has to be reassigned to Northtown!

Example: Garbage Time

Metro Garbage Truck Apportionments

District	Homes	Quota	Apportionment
Northtown	10,450	10.45	10
Southtown	89,550	89.55	90
Total	100,000	100.00	100

Revised Metro Garbage Truck Apportionments

District	Homes	Quota	Apportionment
Northtown	10,450	10.42	11
Southtown	89,550	89.34	89
Newtown	5,250	5.24	5
Total	105,250	105.00	105

Notice that the standard divisor has gone up a little and is now approximately 1002.38.

Hamilton's Method

There are two key lessons we should take from this section:

- (1) In terms of fairness, Hamilton's method leaves a lot to be desired; and
- (2) the critical flaw in Hamilton's method is the way it handles the surplus seats. Clearly, there must be a better apportionment method.

Jefferson's Method

Jefferson's method is based on an approach very different from Hamilton's method. It is in the handling of the surplus seats (Step 3) that Hamilton's method runs into trouble.

In Jefferson's method, when the quotas are rounded down, there are no surplus seats.

This is achieved by changing the divisor, which then changes the quotas. The idea is that by using a smaller divisor, we make the quotas bigger.

Example: Parador's Congress (Jefferson's Method)

When we divide the populations by the standard divisor $SD = 50,000$, we get the standard quotas (i.e in Hamilton's method). When these quotas are rounded down, we end up with four surplus seats.

In the Jefferson's method we have some new calculations.

Example: Parador's Congress (Jefferson's Method)

When we divide the populations by the slightly smaller divisor $D = 49,500$, we get a slightly bigger set of quotas (fifth row). When these *modified quotas* are rounded down (last row), the surplus seats are gone and we have a valid apportionment.

Parador's Congress: Jefferson Apportionment

State	A	B	C	D	E	F	Total
Population	1,646,000	6,936,000	154,000	2,091,000	685,000	988,000	12,500,000
Standard quota ($SD = 50,000$)	32.92	138.72	3.08	41.82	13.70	19.76	250.00
Lower quota	32	138	3	41	13	19	246
Modified quota ($D = 49,500$)	33.25	140.12	3.11	42.24	13.84	19.96	
Lower quota	33	140	3	42	13	19	250

Of course, you are wondering, where did that 49,500 come from?

Apportionment and Standard Divisor

The example illustrates a key point: Apportionments don't have to be based exclusively on the standard divisor.

Jefferson's method is but one of a group of apportionment methods based on the principle that the standard yardstick 1 seat = SD of people is not set in concrete and that,

if necessary, we can change to a different yardstick: 1 seat = D people, where D is a suitably chosen number.

Modified Divisor and Divisor Methods

The number D is called a **divisor** (sometimes we use the term modified divisor), and apportionment methods that use modified divisors are called **divisor methods**.

Different divisor methods are based on different philosophies of how the **modified quotas** *should be rounded to whole numbers*, but they all follow one script:

When you are done rounding the modified quotas, all M seats have been apportioned (no more, and no less). To be able to do this, **you just need to find a suitable divisor D** .

JEFFERSON'S METHOD

- Step 1** Find a “suitable” divisor D .
- Step 2** Using D as the divisor, compute each state's modified quota (modified quota = state population/ D).
- Step 3** Each state is apportioned its modified *lower quota*.

Finding D

How does one find a suitable divisor D ? There are different ways to do it, we will use a basic, blue-collar approach: educated trial and error.

Our target is a set of modified lower quotas whose sum is M . For the sum of the modified lower quotas to equal M , we need to make the modified quotas somewhat bigger than the standard quotas.

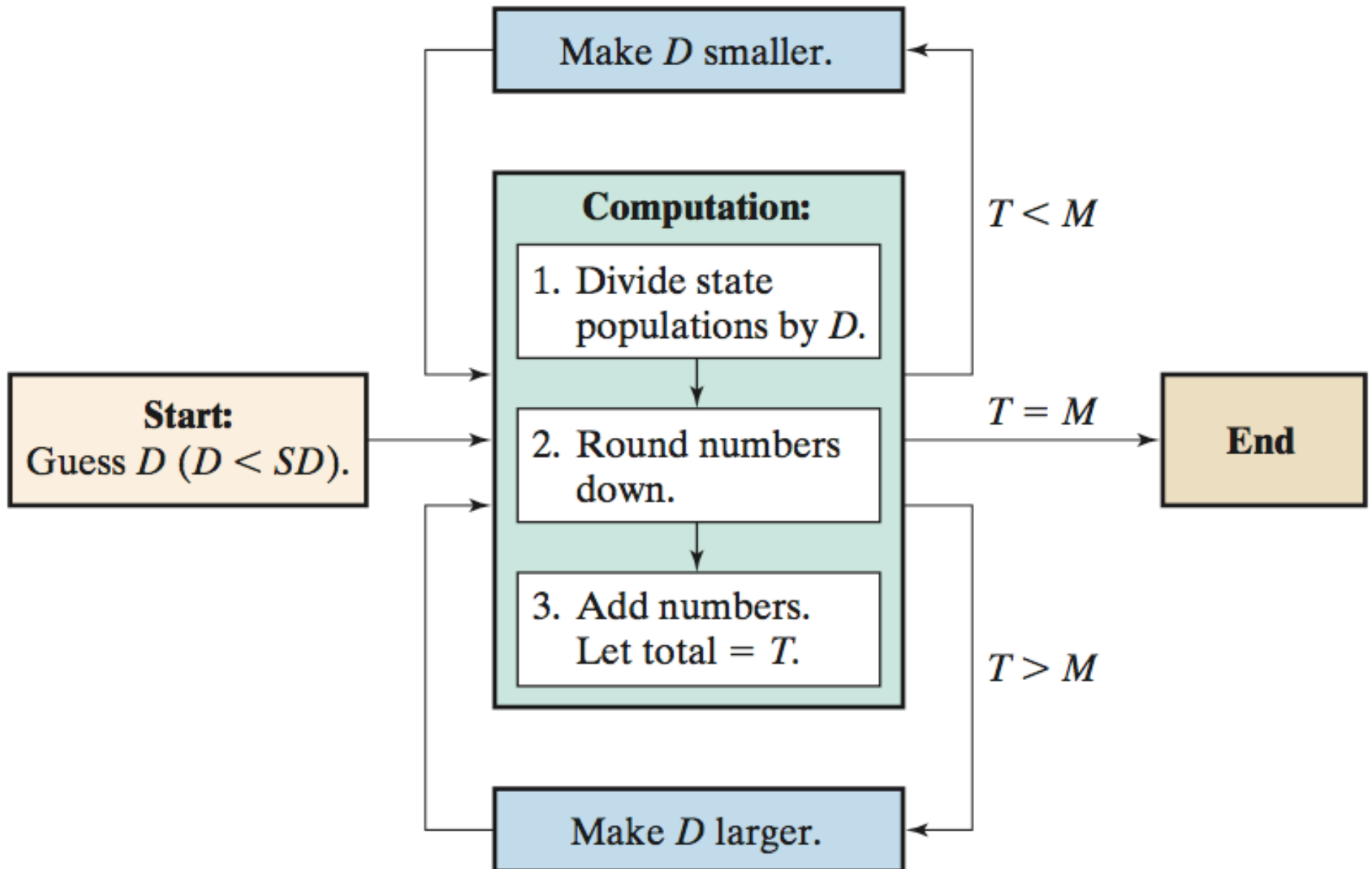
This can only be accomplished by choosing a divisor somewhat smaller than SD . (As the divisor goes down, the quota goes up, and vice versa.)

Finding D

- Make an educated guess, choose a divisor D smaller than SD .
- If guess works, we're done.
- If sum of lower quotas is less than M , *choose even smaller value for D .*
- If sum of lower quotas is more than M , *choose a bigger value for D .*
- Repeat this trial-and-error approach to find a divisor D that works.

Here's a flow chart

Finding D



Example: Parador's Congress (Jefferson's Method)

The first guess should be a divisor somewhat smaller than $SD = 50,000$. Start with $D = 49,000$.

Using this divisor, we calculate the quotas, round them down, and add. We get a total of $T = 252$ seats. We overshot our target by two seats!

Refine our guess by choosing a larger divisor D (the point is to make the quotas smaller). A reasonable next guess (halfway between 50,000 and 49,000) is 49,500. We go through the computation, and it works!

Jefferson's Method and Quota Rule

By design Jefferson's method is consistent in its approach to apportionment—the same formula applies to every state (find a suitable divisor, find the quota, round it down).

By doing this, Jefferson's method is able to avoid the major pitfalls of Hamilton's method.

There is, however, *a serious problem with Jefferson's method that can surface when we least expect it—it can produce upper-quota violations!*—which tend to consistently favor the larger states.

Adams's Method

Like Jefferson's method, Adams's method is a divisor method, but instead of rounding the quotas down, it rounds them up.

For this to work the modified quotas have to be made *smaller*, and this requires the use of a divisor D larger than the standard divisor SD .

ADAMS' S METHOD

- Step 1** Find a “suitable” divisor D .
- Step 2** Using D as the divisor, compute each state's modified quota (modified quota = state population/ D).
- Step 3** Each state is apportioned its modified *upper quota*.

Example: Parador's Congress (Adams's Method)

We know that in Adams's method, the modified divisor D will have to be bigger than the standard divisor of 50,000. We start with the guess $D = 50,500$. The total is $T = 251$, one seat above our target of 250, so we need to make the quotas just a tad smaller.

Example: Parador's Congress (Adams's Method)

We know that in Adams's method, the modified divisor D will have to be bigger than the standard divisor of 50,000. We start with the guess $D = 50,500$. The total is $T = 251$, one seat above our target of 250, so we need to make the quotas just a tad smaller.

Parador's Congress: Adams' Apportionment

State	A	B	C	D	E	F	Total
Population	1,646,000	6,936,000	154,000	2,091,000	685,000	988,000	12,500,000
Modified quota 1 ($D = 50,500$)	32.59	137.35	3.05	41.41	13.56	19.56	
Upper quota	33	138	4	42	14	20	251
Modified quota 2 ($D = 50,700$)	32.47	136.80	3.04	41.24	13.51	19.49	
Upper quota	33	137	4	42	14	20	250

Increase the divisor a little bit, try $D = 50,700$. This divisor works! The apportionment under Adams's method is shown in the last row.

Adams's Method and the Quota Rule

The example highlights a serious weakness of this method—it can produce lower-quota violations!

This is a different kind of violation, but just as serious as the one in Jefferson's method where state B got 1.72 fewer seats than what it rightfully deserves!

We can reasonably conclude that Adams's method is no better (or worse) than Jefferson's method—just different.

Webster's Method

What is the obvious compromise between rounding all the quotas down (Jefferson's method) and rounding all the quotas up (Adams' method)?

What about conventional rounding (Round the quotas down when the fractional part is less than 0.5 and up otherwise.)?

Now that we know that we can use *modified divisors* to manipulate the quotas, it is always possible to find a suitable divisor that will make conventional rounding work. This is the idea behind Webster's method.

WEBSTER'S METHOD

- Step 1** Find a “suitable” divisor D .
- Step 2** Using D as the divisor, compute each state's modified quota (modified quota = state population/ D).
- Step 3** Find the apportionment by rounding each modified quota the conventional way.

Example: Parador's Congress (Webster's Method)

Our first decision is to make a guess at the divisor D :

Use the standard quotas as a starting point.

When we round off the standard quotas to the nearest integer, we get a total of 251. (row 4 of Table on the next slide).

This number is too high (just by one seat), which tells us that we should try a divisor D a tad larger than the standard divisor.

We try $D = 50,100$.

Example: Parador's Congress (Webster's Method)

Parador's Congress: Webster's Apportionment

State	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	Total
Population	1,646,000	6,936,000	154,000	2,091,000	685,000	988,000	12,500,000
Standard quota ($SD = 50,000$)	32.92	138.72	3.08	41.82	13.70	19.76	250.00
Nearest integer	33	139	3	42	14	20	251
Modified quota ($D = 50,100$)	32.85	138.44	3.07	41.74	13.67	19.72	
Nearest integer	33	138	3	42	14	20	250

Row 5 shows the respective modified quotas, and the last row shows these quotas rounded to the nearest integer. Now we have a valid apportionment! The last row shows the final apportionment under Webster's method.

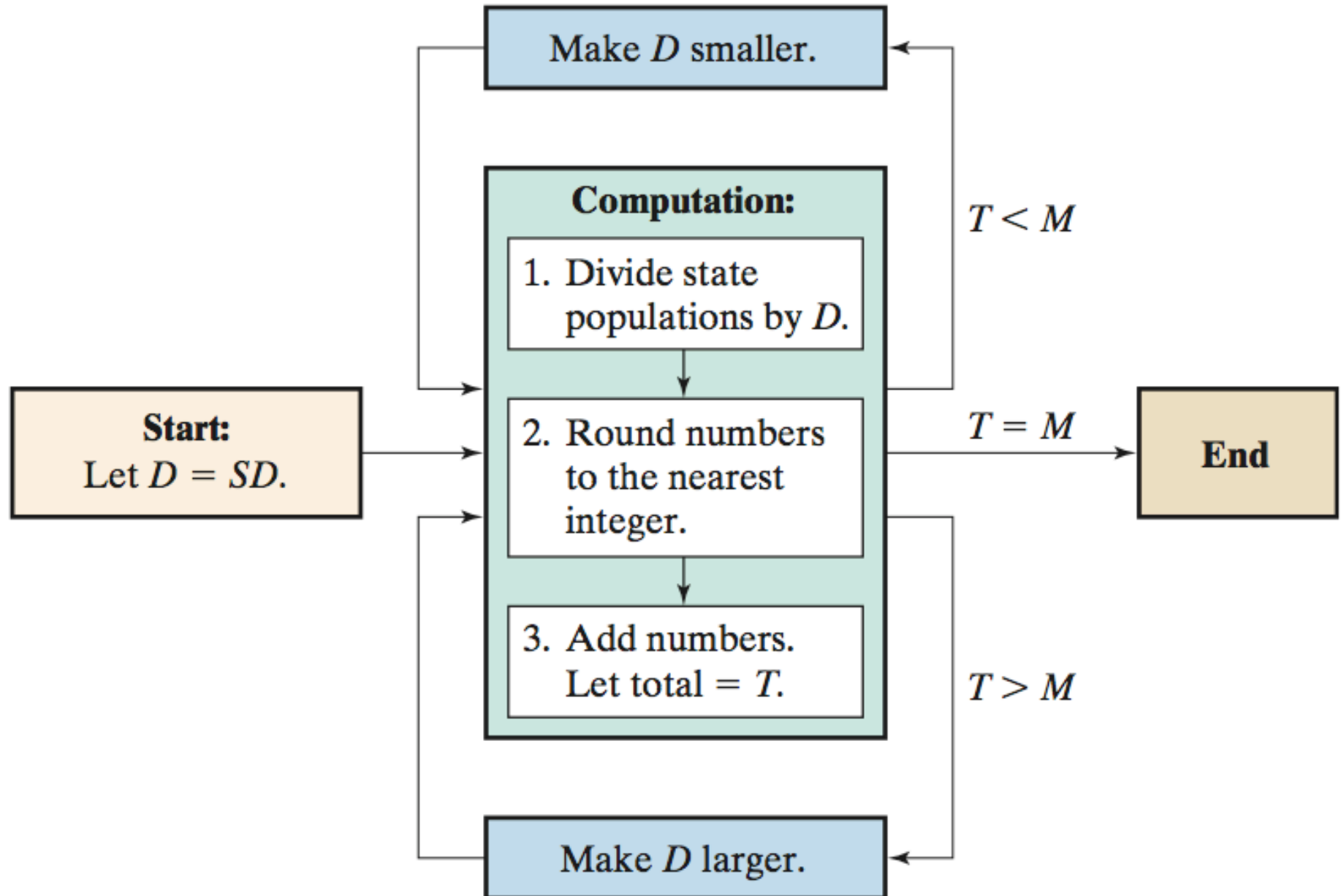
Webster's Method

A flowchart illustrating how to find a suitable divisor D for Webster's method using educated trial and error is on the next slide.

The most significant difference when we use trial and error to implement Webster's method as opposed to Jefferson's method is the choice of the starting value of D .

With Webster's method we always start with the standard divisor SD . If we are lucky and SD happens to work, we are done!

Webster's Method



Webster's Method

When the standard divisor works as a suitable divisor for Webster's method, every state gets an apportionment that is within 0.5 of its standard quota.

This is as good an apportionment as one can hope for. If the standard divisor doesn't quite do the job, there will be at least one state with an apportionment that differs by more than 0.5 from its standard quota.

Webster's Method

In general, Webster's method tends to produce apportionments that don't stray too far from the standard quotas, although occasional violations of the quota rule (both lower- and upper-quota violations), are possible.

Such violations are rare in real-life apportionments. Webster's method has a lot going for it—

it does not suffer from any paradoxes, and it shows no bias between small and large states.